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Number Theory

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*More Pythagorean Triples and the Unit Circle:*

*Derivation and Numerical Analysis*

The Unit Circle and Its Relation to Pythagorean Triples

The Pythagorean Theorem, has been shown to have many incredible properties. One of which is as follows:

Suppose , is divided off by One then obtains the equation:

Define these pair of numbers, as and respectively. Therefore:

This is indeed the unit circle, a circle centered about the origin with a radius of 1. The circle has several obvious points that lie on the axis of .



Suppose we take any given line that goes through the point and has an equation of

. It is obvious then that line will indeed intersect the circle at some other point.

Through the substitution of the derived equations one can find this intersection point:

It is known that equation x + 1 = 0 is a solution to the intersections of the unit circle and our line by definition. Therefore, using long division, one can obtain the other root as follows:

Therefore, the other solution to our circle equation is the point:

For any rational number . All rational numbers, by definition, can be written as a ratio of two integers. Suppose . Therefore, our point becomes as follows:

Now, we notice that we have an ordered point similar of the form

Multiplying out by the denominator we can see that:

(,

Giving us all possibilities of Pythagorean triples.

Thus, all points that lie on the unit circle, are of the form:

Quite a remarkable conclusion.

Q.E.D.

3.1) On Triples of the form (,

Notice that if , as

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Q.E.D.

Though having a common factor excludes the triple from being primitive, a non-primitive triple of this form will not always come directly as a result of having

For example, suppose . This would yield the triple which is reducible over the integers, and therefore not primitive.

As already proven, if the triple using the derived equation will not be primitive. Thought, not all non-primitive triples meet this requirement as shown, for example, . It is easy to show that this second case only arises when the parity of are different That is to say that . This is trivial to show:

If you do suppose that the parities of are different and that the

As show,

As shown,

Q.E.D.

Take the line, We can rearrange this into . Substituting for the y of the circle equation, we get

Subtract by 2 and divide by

Doing long division by , yields

It is worth noting that this cannot be done for the equation , as does not divide properly.

However, Solving for x,

Substituting for x in the line equation we get

Therefore, the rational solutions to the equation are:

Now notice that,

3.3) The Hyperbola

The same methods above will be done again here:

Dividing by the root will give us the other solution,

Therefore, the equation other point on the line is

Now notice,

Thus, we have again found the ration of the the triple embedded in the rational coordinates of a line intersecting a polynomial. It does appear that the rational points of these intersecting lines correspond to the 3-term arithmetic progression of squares quite often.

Q.E.D.

Revisiting Triangular Squares

As shown, a triangular number can be generated with the following equation first described by Gauss as a child,

These numbers are indeed rare, and they represent the numbers that can be geometrically transformed into a triangle of dots. In fact, some numbers are indeed both perfect squares and triangular numbers. For example, 36.

Completing the square yields,

This equation can indeed be represented by the Pell equation of,

The first several computed solutions to this equation is: (17, 12), (99, 70), (577, 408) corresponding to the triangular square numbers 1, 36, 1225, 41616, 1413721, 48024900.

Now take the line, and solve the solution to the equation, getting all possible rational values that lie on the Pell equation:

Long division with the root of will yield a new equation where the other x value, or the other intersection to the line can be found.

Solving for in the line equation will yield a variable form of all rational coordinates lying on the Pell equation.